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Kolmogorov widths between the anisotropic space and the space of functions with mixed smoothness

Heping Wang[a,*,1] and Yongsheng Sun[b,2]

[a] Department of Mathematics, Capital Normal University, Beijing 100037, People's Republic of China
[b] Department of Mathematics, Beijing Normal University, Beijing 100875, People's Republic of China

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Abstract

The Kolmogorov width of the classes of functions with mixed smoothness in the anisotropic space and that of the anisotropic class in the space of functions with mixed smoothness are considered. The asymptotic order of the widths and weakly asymptotic optimal approximation subspaces which realize the order of widths are also given.

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1. Introduction

Let $\pi_d = [0, 2\pi]^d$ be a d-dimensional torus, and let $L_p := L_p(\pi_d)$, $1 \leqslant p \leqslant \infty$, be the usual space of L_p-integrable functions on π_d. For $e \subset e_d := \{1, 2, \ldots, d\}$, $r = (r_1, \ldots, r_d) > 0$ (i.e., $r_j > 0, j = 1, \ldots, d$), let $D^{r^e} f(x) := (\prod_{j \in e} \frac{\partial^{r_j}}{\partial x_j^{r_j}}) f(x)$ be the generalized

*Corresponding author.
E-mail addresses: wanghp@mail.cnu.edu.cn (H. Wang), sunys@bnu.edu.cn (Y. Sun).
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derivative of f in the sense of Weyl (see [2,4]). In the following, we always suppose that $r=(r_1,\dots,r_d)\in\mathbb{R}^d, \mathbf{R}=(R_1,\dots,R_d)\in\mathbb{R}^d$, r, $\mathbf{R}>0$. Then the anisotropic Sobolev space $W_p^{\mathbf{R}}$ and the Sobolev space $W_{\mathrm{mix},\,p}^r$ of functions with mixed derivative are defined as follows (see [4]):

$$W_p^{\mathbf{R}} := \left\{ f\in L_p : \|f\|_{W_p^{\mathbf{R}}} := \|f\|_p + \sum_{j=1}^{d} \left\| \frac{\partial^{R_j} f}{\partial x_j^{R_j}} \right\|_p < \infty \right\}, \tag{1.1}$$

$$W_{\mathrm{mix},\,p}^r := \left\{ f\in L_p :\ \|f\|_{W_{\mathrm{mix},\,p}^r} := \sum_{e\subset e_d} \|D^{r^e} f\|_p < \infty \right\}. \tag{1.2}$$

When $R_1=R_2=\cdots=R_d$, $W_p^{\mathbf{R}}$ is the usual Sobolev space.

For $\mathbf{R}$, $r>0$, we denote $k_i=[R_i]+1,\ l_i=[r_i]+1, 1\leqslant i\leqslant d$. Then the anisotropic Hölder–Nikolskii space $H_p^{\mathbf{R}}$ and the Hölder–Nikolskii space $H_{\mathrm{mix},\,p}^r$ of functions with mixed difference are defined in the following way (see [4]):

$$H_p^{\mathbf{R}} := \left\{ f\in L_p : \|f\|_{H_p^{\mathbf{R}}} := \|f\|_p + \sum_{j=1}^{d} \sup_{h_j>0} h_j^{-R_j}\cdot \|\Delta_{h_j,j}^{k_j} f\|_p < \infty \right\}, \tag{1.3}$$

$$H_{\mathrm{mix},\,p}^r := \left\{ f\in L_p : \|f\|_{H_{\mathrm{mix},\,p}^r} := \sum_{e\subset e_d} \sup_{h>0} \prod_{j\in e} h_j^{-r_j}\cdot \|\Delta_{h^e}^{l^e} f\|_p < \infty \right\}, \tag{1.4}$$

where $h=(h_1,\dots,h_d)>0$, and

$$\Delta_{h^e}^{l^e} f(x) := \left(\prod_{j\in e} \Delta_{h_j,j}^{l_j} \right) f(x), \quad \Delta_{h_j,j}^{l_j} f(x) := \sum_{k=0}^{l_j} (-1)^{l_j-k} \binom{l_j}{k} f(x_1,\dots,x_j+kh_j,\dots,x_d).$$

Let X be a Banach space with the norm $\|\cdot\|_X$, and F be a (convex, compact, centrally symmetric) subset of X. Then the Kolmogorov M-width of F in X is defined by

$$d_M(F,X) := \inf_{L_M} \sup_{f\in F} \inf_{g\in L_M} \|f-g\|_X, \tag{1.5}$$

where L_M runs over all subspaces of X of dimension M or less. More information on Kolmogorov widths can be found in [3].

Let $F_{\mathrm{mix},p}^r$ denote one of the spaces with mixed smoothness $W_{\mathrm{mix},p}^r$, $H_{\mathrm{mix},p}^r$, let $F_p^{\mathbf{R}}$ denote one of the anisotropic spaces $W_p^{\mathbf{R}}$, $H_p^{\mathbf{R}}$, and let $BF_{\mathrm{mix},p}^r$, $BF_p^{\mathbf{R}}$ be the unit balls of the spaces $F_{\mathrm{mix},p}^r$, $F_p^{\mathbf{R}}$, respectively. In this paper, we consider the Kolmogorov M-width of the classes $BF_{\mathrm{mix},p}^r$ in the space $F_p^{\mathbf{R}}$ (if $r-\mathbf{R}>0$) and that of the classes $BF_p^{\mathbf{R}}$ in the space $F_{\mathrm{mix},p}^r$ (if $\sum_{i=1}^d r_i/R_i<1$). We find the asymptotic order of the widths and give weakly asymptotic optimal approximation subspaces which realize the order of widths. Our main results are the following:

Theorem 1. *Let* $1\leqslant p\leqslant\infty$, $r-\mathbf{R}>0$, $\bar{\alpha}:=\min_{1\leqslant i\leqslant d}\alpha_i, \alpha_i=r_i-R_i, i=1,\dots,d$. *Then*

$$d_M(BF^r_{\mathrm{mix},p}, F^{\mathbf{R}}_p)\asymp M^{-\bar{\alpha}}. \tag{1.6}$$

Theorem 2. *Let* $1\leqslant p\leqslant\infty$, $\upsilon:=\sum_{i=1}^d r_i/R_i<1,\ g(\mathbf{R}):=\left(\sum_{j=1}^d R_j^{-1}\right)^{-1}$. *Then*

$$d_M(BF^{\mathbf{R}}_p, F^r_{\mathrm{mix},p})\asymp M^{-(1-\upsilon)g(\mathbf{R})}. \tag{1.7}$$

Remark. In [1] Bugrov obtained that $H^{\mathbf{R}}_p$ can be continuously imbedded into the space $H^r_{\mathrm{mix},\,p}$ if $\sum_{i=1}^d r_i/R_i\leqslant 1$. Here Theorem 2 shows that $F^{\mathbf{R}}_p$ can be compactly imbedded into the space $F^r_{\mathrm{mix},\,p}$ if $\sum_{i=1}^d r_i/R_i<1$. Also Theorem 1 shows that $F^r_{\mathrm{mix},\,p}$ can be compactly imbedded into the space $F^{\mathbf{R}}_p$ if $r-\mathbf{R}>0$.

2. Some lemmas

Lemma 1. *Let* $1\leqslant p\leqslant\infty$, $r,\mathbf{R}>0$. *Then for all* $f\in W^{\mathbf{R}}_p$, $g\in W^r_{\mathrm{mix},p}$, *we have*

$$||f||_{H^{\mathbf{R}}_p}\ll||f||_{W^{\mathbf{R}}_p};\quad ||g||_{H^r_{\mathrm{mix},p}}\ll||g||_{W^r_{\mathrm{mix},p}}. \tag{2.1}$$

Let $V_m(t)$ be the Vallee–Poussin kernel on $[0,2\pi]$; that is

$$V_m(t)=1+2\sum_{k=1}^m\cos kt+2\sum_{k=m+1}^{2m-1}\left(1-\frac{k-m}{m}\right)\cos kt;\quad V_{2^{-1}}(t)\equiv 1,\ V_{2^{-2}}(t)\equiv 0.$$

For $f\in L_p$, $s=(s_1,\dots,s_d)\in\mathbb{Z}^d$, $s\geqslant 0$, we define

$$A_s(x)=\prod_{j=1}^d(V_{2^{s_j-1}}(x_j)-V_{2^{s_j-2}}(x_j)),\quad A_sf(x)=A_s*f(x). \tag{2.2}$$

Lemma 2. *Let* $1\leqslant p\leqslant\infty$, $r=(r_1,\dots,r_d)>0$. *Then for any* $f\in H^r_{\mathrm{mix},p}$, *we have*

$$||f||_{H^r_{\mathrm{mix},p}}\asymp\sup_{s\geqslant 0}2^{(r,s)}||A_sf||_p. \tag{2.3}$$

where $(r,s)=r_1s_1+\cdots+r_ds_d$.

For $\mathbf{R}=(R_1,\dots,R_d)>0$, $s=(s_1,\dots,s_d)\in\mathbb{Z}^d$, $s\geqslant 0$, $n\in\mathbb{Z}_+$, we denote

$$g(\mathbf{R}):=\left(\sum_{j=1}^d R_j^{-1}\right)^{-1},\quad Q^n_{\mathbf{R}}:=\{s\geqslant 0\,|\,s_j\leqslant[ng(\mathbf{R})/R_j],\ 1\leqslant j\leqslant d\}, \tag{2.4}$$

$$V(f,\mathbf{R},n)(x) = \sum_{s\in Q_{\mathbf{R}}^n} A_s f(x), \tag{2.5}$$

$$A(f,\mathbf{R},0) = V(f,\mathbf{R},0); \quad A(f,\mathbf{R},n) = V(f,\mathbf{R},n) - V(f,\mathbf{R},n-1). \tag{2.6}$$

Lemma 3. *Let* $1 \leqslant p \leqslant \infty$, $\mathbf{R} = (R_1,\dots,R_d) > 0$. *Then for any* $f \in H_p^{\mathbf{R}}$, *we have*

$$\|f\|_{H_p^{\mathbf{R}}} \asymp \sup_{n\geqslant 0} 2^{n\,g(\mathbf{R})} \|A(f,\mathbf{R},n)\|_p \tag{2.7}$$

For the proof of Lemmas 1–3, see [4].

First we suppose that $\upsilon := \sum_{i=1}^d r_i/R_i < 1$. For $J \in \mathbb{Z}_+$, we denote

$$\Psi_{\mathbf{R}}^J := \bigcup_{s\in Q_{\mathbf{R}}^J} \rho(s), \quad \rho(s) := \{k = (k_1,\dots,k_d) \in \mathbb{Z}^d | [2^{s_j-1}] \leqslant |k_j| < 2^{s_j},\ j = 1,\dots,d\}$$

$$\delta_s f(x) := \sum_{k\in\rho(s)} \hat{f}(k) e^{i(k,x)}, \quad \hat{f}(k) := (2\pi)^{-d} \int_{\pi_d} f(x) e^{-i(k,x)}\,dx,$$

$$T(\Psi_{\mathbf{R}}^J)_p := \left\{ f \in L_p | f(x) = \sum_{k\in\Psi_{\mathbf{R}}^J} \hat{f}(k) e^{ikx} = \sum_{s\in Q_{\mathbf{R}}^J} \delta_s f(x) \right\}. \tag{2.8}$$

It is easy to show that $V(f,\mathbf{R},J)$ is a bounded linear operator from L_p into $T(\Psi_{\mathbf{R}}^J)_p$ $(1 \leqslant p \leqslant \infty)$ and dim $T(\Psi_{\mathbf{R}}^J)_p \asymp 2^J$. Moreover, we have

Lemma 4. *Let* $1 \leqslant p \leqslant \infty$, $\upsilon = \sum_{i=1}^d r_i/R_i < 1$. Then for any $f \in BH_p^{\mathbf{R}}$, we have

$$\|f - V(f,\mathbf{R},J)\|_{W_{\mathrm{mix},\,p}^r} \ll 2^{-(1-\upsilon)Jg(\mathbf{R})}. \tag{2.9}$$

Proof. By the Bernstein inequality and Lemma 3, we have

$$\|f - V(f,\mathbf{R},J)\|_{W_{\mathrm{mix},p}^r} = \left\| \sum_{n>J} A(f,\mathbf{R},n) \right\|_{W_{\mathrm{mix},p}^r}$$

$$\leqslant \sum_{n>J} \|A(f,\mathbf{R},n)\|_{W_{\mathrm{mix},p}^r} \ll \sum_{n>J} 2^{\sum_{i=1}^d r_i[n\,g(\mathbf{R})/R_i]} \|A(f,\mathbf{R},n)\|_p$$

$$\ll \sum_{n>J} 2^{\upsilon\,n\,g(\mathbf{R})} 2^{-n\,g(\mathbf{R})} \|f\|_{H_p^{\mathbf{R}}} \ll 2^{-(1-\upsilon)Jg(\mathbf{R})} \tag{2.10}$$

Lemma 4 is proved. □

For $J \in Z_+$, $\bar{s} := ([Jg(\mathbf{R})/R_1], \ldots, [Jg(\mathbf{R})/R_d])$. Denote

$$T(\bar{s})_p := \left\{ f \in L_p | f(x) = \sum_{k \in \rho(\bar{s})} \hat{f}(k) e^{ikx} = \delta_{\bar{s}} f(x) \right\}.$$

Lemma 5. *Let* $1 \leqslant p \leqslant \infty$, $\upsilon = \sum_{i=1}^{d} r_i/R_i$, $f \in T(\bar{s})_p$. *Then*

$$||f||_{H^r_{\mathrm{mix},p}} \geqslant 2^{\upsilon g(\mathbf{R})J} ||f||_p. \tag{2.11}$$

Proof. Denote $Q_{\bar{s}} = \{s = (s_1, \ldots, s_d) |\, s_i = [Jg(\mathbf{R})/R_i]$ or $[Jg(\mathbf{R})/R_i] + 1,\ i = 1, \ldots, d\}$. It is easy to verify that $A_s f \equiv 0$ if $s \bar{\in} Q_{\bar{s}}$ and the number of the element of $Q_{\bar{s}}$ is 2^d. By Lemma 2 we get that

$$\begin{aligned} ||f||_{H^r_{\mathrm{mix},p}} &\asymp \sup_{s \geqslant 0} 2^{(r,s)} ||A_s f||_p \asymp 2^{\upsilon g(\mathbf{R})J} \max_{s \in Q_{\bar{s}}} ||A_s f||_p \\ &\asymp 2^{\upsilon g(\mathbf{R})J} \sum_{s \in Q_{\bar{s}}} ||A_s f||_p \geqslant 2^{\upsilon g(\mathbf{R})J} || \sum_{s \in Q_{\bar{s}}} A_s f||_p = 2^{\upsilon g(\mathbf{R})J} ||f||_p. \end{aligned}$$

Now we consider the case $\alpha := r - \mathbf{R} > 0$. For $J \in \mathbb{Z}_+$, $\bar{\alpha} := \min_{1 \leqslant i \leqslant d} \alpha_i$, $\alpha_i = r_i - R_i$, $0 < \mu < 1$, we define

$$\Psi^\mu_J := \bigcup_{s \in Q^\mu_J} \rho(s), \quad Q^\mu_J := \left\{ s \in \mathbb{Z}^d_+ |\ (\alpha, s) + \mu \left((\mathbf{R}, s) - \max_{1 \leqslant i \leqslant d} (R_i s_i) \right) \leqslant \bar{\alpha} J \right\} \tag{2.12}$$

$$T(\Psi^\mu_J)_p := \left\{ f \in L_p || f(x) = \sum_{k \in \Psi^\mu_J} \hat{f}(k) e^{ikx} = \sum_{s \in Q^\mu_J} \delta_s f(x) \right\}, \quad \Box \tag{2.13}$$

$T(\Psi^\mu_J)_p$ is called hyperbolic cross trigonometric subspace. We want to compute its dimension. First we introduce a lemma.

Lemma 6. (see Temlykov [4]) *Let* $\kappa > 0$, $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\beta = (\beta_1, \ldots, \beta_d) > 0$, $\gamma_i = \alpha_i/\beta_i$, $i = 1, \ldots, d$, and $\gamma = \gamma_1 = \cdots = \gamma_v < \gamma_{v+1} \leqslant \cdots \leqslant \gamma_d$, $1 \leqslant v \leqslant d$. *Then*

$$\sum_{(\beta,s) > J} 2^{-\kappa(\alpha,s)} \asymp 2^{-\kappa\gamma J} \cdot J^{v-1} \quad \text{and} \quad \sum_{(\alpha,s) \leqslant J} 2^{\kappa(\beta,s)} \asymp 2^{\kappa\gamma J} \cdot J^{v-1}. \tag{2.14}$$

From Lemma 6 we know that

$$\dim T(\Psi^\mu_J)_p = |\Psi^\mu_J| = \sum_{s \in Q^\mu_J} |\rho(s)| \asymp \sum_{s \in Q^\mu_J} 2^{(s,\omega)} \asymp \sum_{j=1}^{d} \sum_{(s,\alpha^j) \leqslant \bar{\alpha} J} 2^{(s,\omega)} \asymp 2^J, \tag{2.15}$$

where $\omega = (1, \ldots, 1)$, $\alpha^j = (\alpha^j_1, \ldots, \alpha^j_d)$, $\alpha^j_j = \alpha_j$; $\alpha^j_i = \alpha_i + \mu R_i$, $i \neq j$.

For $f \in L_p$, we set

$$V_J^\mu(f,x) = \sum_{s \in Q_J^\mu} A_s f(x). \tag{2.16}$$

Then V_J^μ is a bounded linear operator from L_p into $T(\Psi_J^\mu)_p$ $(1 \leqslant p \leqslant \infty)$. Moreover, we have

Lemma 7. *Let* $1 \leqslant p \leqslant \infty,\ 0 < \mu < 1,\ r - \mathbf{R} > 0,\ \bar{\alpha} := \min_{1 \leqslant i \leqslant d} \alpha_i,\ \alpha_i = r_i - R_i$. *Then for any* $f \in BH^r_{\mathrm{mix},p}$, *we have*

$$\|f - V_J^\mu f\|_{W_p^{\mathbf{R}}} \ll 2^{-\bar{\alpha}J}. \tag{2.17}$$

Proof. By the Bernstein inequality and Lemma 2, we have

$$\|f - V_J^\mu f\|_{W_p^{\mathbf{R}}} = \left\| \sum_{s \bar{\in} Q_J^\mu} A_s f \right\|_{W_p^{\mathbf{R}}}$$

$$\leqslant \sum_{s \bar{\in} Q_J^\mu} \|A_s f\|_{W_p^{\mathbf{R}}} \ll \sum_{s \bar{\in} Q_J^\mu} 2^{\max_{1 \leqslant i \leqslant d}(R_i s_i)} \|A_s f\|_p$$

$$\ll \sum_{s \bar{\in} Q_J^\mu} 2^{-(r,s) + \max_{1 \leqslant i \leqslant d}(R_i s_i)} \|f\|_{H^r_{\mathrm{mix},p}} \leqslant K \tag{2.18}$$

where $K := \sum_{s \bar{\in} Q_J^\mu} 2^{-(r,s)+\max_{1 \leqslant i \leqslant d}(R_i s_i)}$. Let us estimate K. Since $0 < \mu < 1$, $\frac{\alpha_i + R_i}{a_i + \mu R_i} > 1$, by Lemma 6 we know that

$$K = \sum_{s \bar{\in} Q_J^\mu} 2^{-(r,s)+\max_{1 \leqslant i \leqslant d}(R_i s_i)} \leqslant \sum_{j=1}^{d} \sum_{(s,\alpha^j) > \bar{\alpha}J} 2^{-(s,\beta^j)} \ll 2^{-\bar{\alpha}J}$$

where $\beta^j := (\beta_1^j, \ldots, \beta_d^j)$, $\alpha^j := (\alpha_1^j, \ldots, \alpha_d^j)$, $\alpha_j^j = \beta_j^j = \alpha_j$, $\alpha_i^j = \alpha_i + \mu R_i$, $\beta_i^j = \alpha_i + R_i$, $i \neq j$.

Lemma 7 is proved. □

Lemma 8. (see Pinkus [3]). *Let X_n be a subspace of a normed linear space X whose dimension n is greater than M, and let BX_n denote the closed unit ball of X_n. Then*

$$d_M(BX_n, X) = 1.$$

3. Proof of Theorems 1 and 2

First, by (2.1) we know that

$$d_M(BW^r_{\mathrm{mix},p}, H^{\mathbf{R}}_p) \ll \begin{matrix} d_M(BW^r_{\mathrm{mix},p}, W^{\mathbf{R}}_p) \\ d_M(BH^r_{\mathrm{mix},p}, H^{\mathbf{R}}_p) \end{matrix} \ll d_M(BH^r_{\mathrm{mix},p}, W^{\mathbf{R}}_p), \tag{3.1}$$

$$d_M(BW^{\mathbf{R}}_p, H^r_{\mathrm{mix},p}) \ll \begin{matrix} d_M(BW^{\mathbf{R}}_p, W^r_{\mathrm{mix},p}) \\ d_M(BH^{\mathbf{R}}_p, H^r_{\mathrm{mix},p}) \end{matrix} \ll d_M(BH^{\mathbf{R}}_p, W^r_{\mathrm{mix},p}). \tag{3.2}$$

So it suffices to prove the upper estimate for $d_M(BH^r_{\mathrm{mix},p}, W^{\mathbf{R}}_p)$, $d_M(BH^{\mathbf{R}}_p, W^r_{\mathrm{mix},p})$ and the lower estimate for $d_M(BW^r_{\mathrm{mix},p}, H^{\mathbf{R}}_p)$, $d_M(BW^{\mathbf{R}}_p, H^r_{\mathrm{mix},p})$.

Upper estimates. By Lemma 7 we see that

$$d_M(BH^r_{\mathrm{mix},p}, W^{\mathbf{R}}_p) \leqslant \sup_{f \in BH^r_{\mathrm{mix},p}} \|f - V^{\mu}_J f\|_{W^{\mathbf{R}}_p} \ll 2^{-\bar{\alpha}J}. \tag{3.3}$$

where $J = J(M)$ satisfies $J = \sup\{k \in \mathbb{Z}_+ | \dim T(\Psi^{\mu}_k)_p \leqslant M\}$, $0 < \mu < 1$. So from (2.15) we can infer that $2^J \asymp M$. Hence

$$d_M(BH^r_{\mathrm{mix},p}, W^{\mathbf{R}}_p) \ll M^{-\bar{\alpha}}. \tag{3.4}$$

Similarly we can get

$$d_M(BH^{\mathbf{R}}_p, W^r_{\mathrm{mix},p}) \ll M^{-(1-\upsilon)g(\mathbf{R})}. \tag{3.5}$$

Lower estimates. Let J satisfy $2^{J-d-1} \leqslant M < 2^{J-d}$, $\bar{s} := ([Jg(\mathbf{R})/R_1], \dots, [Jg(\mathbf{R})/R_d])$. Denote

$$T(\bar{s})_p := \left\{ f \in L_p | f(x) = \sum_{k \in \rho(\bar{s})} \hat{f}(k) e^{ikx} = \delta_{\bar{s}} f(x) \right\}.$$

For all $f \in T(\bar{s})_p$, by the Bernstein inequality and Lemma 4, we have

$$\|f\|_{W^{\mathbf{R}}_p} \ll 2^{Jg(\mathbf{R})} \|f\|_p \ll 2^{(1-\upsilon)Jg(\mathbf{R})} \|f\|_{H^r_{\mathrm{mix},p}}. \tag{3.6}$$

So we obtain that $c \cdot 2^{-(1-\upsilon)g(\mathbf{R})J} \cdot BH^r_{\mathrm{mix},p} \cap T(\bar{s})_p \subset BW^{\mathbf{R}}_p$, where c is a positive constant. Hence by Lemma 8 we obtain that

$$d_M(BW^{\mathbf{R}}_p, H^r_{\mathrm{mix},p}) \gg 2^{-(1-\upsilon)g(\mathbf{R})J} d_M(BH^r_{\mathrm{mix},p} \cap T(\bar{s})_p, H^r_{\mathrm{mix},p}) \gg 2^{-(1-\upsilon)g(\mathbf{R})J}. \tag{3.7}$$

Now let $r - \mathbf{R} > 0$. For simplicity we suppose that $\bar{\alpha} = \min_{1 \leqslant i \leqslant d}(r_i - R_i) = r_1 - R_1$. Let J satisfy $2^{J-2} \leqslant M < 2^{J-1}$, $\tilde{J} = (J, 0, \dots, 0)$. Denote

$$T(\tilde{s})_p := \{f \in L_p | f(x) = \sum_{k \in \rho(\tilde{s})} \hat{f}(k) e^{ikx} = \delta_{\tilde{s}} f(x)\}.$$

For all $f \in T(\tilde{s})_p$, similar to the proof of (3.6), we can get that

$$\|f\|_{W^r_{\mathrm{mix},p}} \ll 2^{\bar{\alpha}J} \|f\|_{H^{\mathbf{R}}_p}. \tag{3.8}$$

So by Lemma 8 we obtain that

$$d_M(BW^r_{\text{mix},p}, H^{\mathbf{R}}_p) \gg 2^{-\bar{\alpha}J} d_M(BH^{\mathbf{R}}_p \cap T(\tilde{s})_p, H^{\mathbf{R}}_p) \gg 2^{-\bar{\alpha}J}. \tag{3.9}$$

The proof of Theorems 1 and 2 is complete.

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